Asymptotic properties of dynamic multipliers in nonlinear econometric models

Carlo Bianchi and Giorgio Calzolari and Paolo Corsi and Lorenzo Panattoni

IBM Scientific Center, Pisa, Italy

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Asymptotic Properties of Dynamic Multipliers in Nonlinear Econometric Models *

1. *Introduction*

Frequent recourse is made to dynamic multipliers (delay and sustained) of macro-econometric models to analyze the effects of alternative policy actions. Following Goldberger (1964, p. 375), delay multipliers measure the effects of impulse exogenous changes after some periods; they are referred to as delayed multipliers in Chow (1975, p. 107) or interim multipliers in Theil and Boot (1962), Schmidt (1973) and Brissimis and Gill (1978). Sustained multipliers (referred to as intermediate-run multipliers in Chow, 1975, p. 107) measure the effects of changes in the exogenous variables that have persisted over several periods.

In order to know how much reliance can be placed on the results of any specific action, it would be useful if an estimate of their degree of uncertainty could be associated to these dynamic multipliers (see, for example, Fair, 1980).

A mixed analytic and simulation approach to derive the asymptotic standard errors of dynamic multipliers in nonlinear econometric models is proposed.

The main reason for deriving asymptotic statistical properties is that, even with linear models (for this case, see Schmidt, 1973, Brissimis and Gill, 1978, or Gill and Brissimis, 1978), the computation of multipliers involves nonlinear transformations of the structural coefficients; the computation of finite sample moments can thus be rather difficult. Asymptotic normality, however, can be maintained even with nonlinear transformations, provided that these transformations are twice continuously differentiable functions in the endogenous variables, the exogenous variables and the parameters. As these conditions are, usually, largely satisfied in econometrics, it is possible to extend the computation of asymptotic standard errors of dynamic multipliers to a quite general class of nonlinear models.

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This paper shows that the estimated dynamic multipliers are asymptotically normally distributed around their probability limits and shows how a consistent estimate of the asymptotic covariance matrix can be computed using a mixed analytic and simulation method. Furthermore, since the probability limits are generally not equal to the *true* dynamic multipliers, an estimate of the differences (inconsistencies) can be computed using a stochastic simulation (Monte Carlo) method.

These two methods are a straightforward extension to dynamic multipliers of the methods proposed in Bianchi, Calzolari and Corsi (1981) for impact multipliers of nonlinear models.

Applications to some nonlinear macroeconomic models of the Italian, German, and U.S. economies are presented in the last sections of this paper.

2. *Main assumptions*

Let a structural dynamic econometric model be represented by

$$f(y_t, x_t, y_{t-1}, \alpha) = u_t; \qquad t = 1, 2, \ldots, T \tag{2.1}$$

where:

f is an $(m \times 1)$ column vector of functional operators $(f_i, i = 1, 2, \ldots, m)$, twice continuously differentiable with respect to the elements of y_t, x_t, y_{t-1} and α;

y_t, y_{t-1} and x_t are $(m \times 1)$, $(m \times 1)$ and $(n \times 1)$ column vectors of current endogenous, lagged endogenous and fixed exogenous variables at time $t = 1, 2, \ldots, T$ $(y_{i,t}, i = 1, 2, \ldots, m; x_{j,t}, j = 1, 2, \ldots, n)$;

α is the $(s \times 1)$ column vector $(\alpha_k, k = 1, 2, \ldots, s)$ of all the structural coefficients of the model to be estimated (the other known coefficients of the model being included in the functional operators);
u_t is the $(m \times 1)$ column vector of independently and identically distributed structural stochastic disturbances at time t $(u_{i,t}, i = 1, 2, \ldots, m)$; they are assumed to have zero mean and finite covariance matrix.

Basic assumptions for this work are the existence of a vector $\hat{\alpha}$ of consistent estimates of α, the asymptotic normality of $\sqrt{T}(\hat{\alpha} - \alpha) \sim N(0, \Psi)$ and the availability of $\hat{\Psi}$, a consistent estimate [1] of Ψ.

Further assumption, as usual for nonlinear models, is that system (2.1) can be uniquely solved at any period either statically, or dynamically, for any relevant value of the predetermined variables, of the coefficients and of the error terms. If the process which generates the endogenous variables at time t starts at time $t - r$ and goes onwards up to t, we assume to represent the vector of endogenous variables at time t as

$$y_t = y(x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, \alpha, u_t, u_{t-1}, \ldots, u_{t-r}) \tag{2.2}$$

where, due to the assumed nonlinearity of model (2.1), the $(m \times 1)$ vector $y(y_i, i = 1, 2, \ldots, m)$ of functional operators cannot generally be determined exactly, but can be approximated (e.g. by some Newton or Gauss Seidel-type solution method) to any desired degree of accuracy.

It is worth recalling that higher than first order lags of endogenous variables, as well as lagged exogenous, can be easily reconducted to the formulation of system (2.1) by the proper insertion of definitional equations (see Theil and Boot, 1962).

3. *Delay multipliers*

Analogously to the definition of reduced form coefficients and impact multipliers in Dhyrmes (1970, p. 508) and Goldberger (1964, p. 369), delay-r multipliers, related to dynamic simulation from time $t - r$ to t, can be defined as first order derivatives of the conditional expectation of each endogenous variable at time t with respect to each exogenous variable lagged r periods, all other predetermined variables being kept constant. The delay-r multipliers are generally organized in an $(m \times n)$ matrix. However, in this paper, it is more convenient to represent them by

$$\pi_{t,r} = vec\left[\frac{\partial E(y_t \mid x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, \alpha)}{\partial x'_{t-r}}\right] \tag{3.1}$$

which is the $(mn \times 1)$ column vector obtained by stacking the columns of the matrix of delay-r multipliers. Besides the subscript r, which indicates the lag, also the subscript t must be introduced since, in nonlinear models, multipliers change with time.

Introducing the y_t functions and applying the differentiation under integral, it follows that:

$$\pi_{t,r} = vec\, E\left[\frac{\partial y}{\partial x'_{t-r}} \mid x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, \alpha\right] \tag{3.2}$$

where all derivatives are computed at point

$$(x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, \alpha, u_t, u_{t-1}, \ldots, u_{t-r}).$$

[1] This assumption is certainly crucial in the general case of nonlinear dynamic models, but can be maintained as «reasonable» assumption according to Gallant (1977, pp. 73-74) and Hatanaka (1978, fn. 8).

Two additional symbols must be introduced:

$$(3.3)\qquad \bar{\pi}_{t,r} = vec\left\{\left[\frac{\partial y}{\partial x'_{t-r}}\right]_{(x_t,\, x_{t-1},\, \ldots,\, x_{t-r},\, y_{t-r-1},\, \alpha,\, 0,\, 0,\, \ldots,\, 0)}\right\}$$

is the $(mn \times 1)$ column vector of the first order derivatives, computed at the point $(x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, \alpha, 0, 0, \ldots, 0)$ (unknown, since it includes the vector of *true* coefficients α), of the endogenous variables at time t, with respect to the exogenous variables at time $t-r$;

$$(3.4)\qquad \hat{\pi}_{t,r} = vec\left\{\left[\frac{\partial y}{\partial x'_{t-r}}\right]_{(x_t,\, x_{t-1},\, \ldots,\, x_{t-r},\, y_{t-r-1},\, \hat{\alpha},\, 0,\, 0,\, \ldots,\, 0)}\right\}$$

is the $(mn \times 1)$ column vector of the estimated delay-r multipliers. This expression corresponds to the way in which model builders usually compute delay multipliers. The derivatives can be computed analytically for linear models or for simple and small nonlinear models; more usually, they can be computed using finite differences as $\Delta \hat{y}_{i,t}/\Delta x_{j,t-r}$ (see, for example, Evans and Klein, 1968, p. 49, or for the case of impact multipliers, Goldberger, 1970, p. 137).

In any case the computation of delay-r multipliers is usually based on dynamic simulation paths from time $t-r$ to time t; endogenous variables lagged more than r periods (y_{t-r-1}) are treated as fixed (predetermined) and set to their historical values. Delay-q multipliers, if $q < r$, would be based on dynamic simulation path starting from $t-q$ while endogenous variables at time $t-q-1$ would be treated as fixed and set to their historical values. This means that, in the general case, delay-q multipliers would not be computed from the same vector of functions, y, used to define delay-r multipliers. To use the same vector y, a formula like the following should be adopted

$$(3.5)\qquad \hat{\pi}_{t,q} = vec\left\{\left[\frac{\partial y}{\partial x'_{t-r}}\right]_{(x_t,\, x_{t-1},\ldots,\, x_{t-r},\, y_{t-r-1},\, \hat{\alpha},\, 0,\ldots,\, 0,\, \hat{u}_{t-q-1},\ldots,\, \hat{u}_{t-r})}\right\}$$

This creates some troubles when investigating relationships between delay and sustained multipliers; in particular, sustained multipliers will no more be equal to the sum of delay multipliers, as they are for linear models.

It would be possible to overcome this difficulty by adopting a different definition of delay multipliers. For example, they could be all related, whichever the lag, to a dynamic simulation path starting from the beginning of the sample period, $t = 1$ (a definition of this kind would be analogous to the definition of impact multipliers in Howrey and Kelejian, 1971, p. 308). However, this alternative definition has not been adopted in this paper since it seems to us that most model builders, whichever computational method they use, calculate in practice multipliers as indicated by equation (3.4).

Given the assumptions on the f_i functions, also the y_i functions are twice continuously differentiable with respect to the elements of $x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}$ and α. Since $\hat{\alpha}$ is assumed to be a consistent estimate of α, the following holds under regularity conditions:

$$(3.6)\qquad plim\ \hat{\pi}_{t,r} = \bar{\pi}_{t,r}.$$

Furthermore, since $\sqrt{T}(\hat{\alpha}-\alpha)$ is assumed to be asymptotically distributed as multivariate normal $N(0, \Psi)$, then

$$(3.7)\qquad \sqrt{T}(\hat{\pi}_{t,r} - \bar{\pi}_{t,r}) =$$

$$= \sqrt{T}\, vec\left\{\left[\frac{\partial y}{\partial x'_{t-r}}\right]_{(x_t,\, x_{t-1},\, \ldots,\, x_{t-r},\, y_{t-r-1},\, \hat{\alpha},\, 0,\, 0,\, \ldots,\, 0)} - \left[\frac{\partial y}{\partial x'_{t-r}}\right]_{(x_t,\, x_{t-1},\, \ldots,\, x_{t-r},\, y_{t-r-1},\, \alpha,\, 0,\, 0,\, \ldots,\, 0)}\right\}$$

is asymptotically distributed as $N(0, J_{t,r}\Psi J'_{t,r})$, where $J_{t,r}$ is the $(mn \times s)$ matrix of the second order derivatives $(\partial^2 y_i/\partial x_{j,t-r}\, \partial \alpha_k)$ computed at point $(x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, \alpha, 0, 0, \ldots, 0)$ (see Rao, 1973, p. 388, iii).

It must be noted that not only the estimate $\hat{\pi}_{t,r}$, but also $\bar{\pi}_{t,r}$ and $\pi_{t,r}$ are random vectors. They all are, in fact, functions of the lagged endogenous variables y_{t-r-1}. Therefore, the results discussed in this section must be interpreted with some care. First of all, equation (3.6) states the convergence in probability not to a constant, but to a random variable, as in Rao (1973, p. 124, xiii). On the other side, the asymptotic distribution derived after equation (3.7) is conditional on a particular value given to the random vector y_{t-r-1}. In all the experiments of this paper, the derived estimates are conditional on the historical value of the lagged endogenous variables (this is the usual way in which delay multipliers are computed).

Computing the second order derivatives $(\partial^2 y_i/\partial x_{j,t-r}\, \partial \alpha_k)$ at point $(x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, \hat{\alpha}, 0, 0, \ldots, 0)$, $\hat{J}_{t,r}$, a consistent estimate of $J_{t,r}$, can be obtained; the square roots of the diagonal elements of $(\hat{J}_{t,r}\hat{\Psi}\hat{J}'_{t,r})/T$ are the estimated asymptotic standard errors of the elements of $\hat{\pi}_{t,r}$.

In nonlinear models, generally $\bar{\pi}_{t,r} \neq \pi_{t,r}$ (see Howrey and Kelejian, 1971, p. 308, for the case of impact multipliers, even if, as already mentioned, a slightly different definition of the multipliers is there adopted). Together with equation (3.6), this inequality states the inconsistency of $\hat{\pi}_{t,r}$ as an estimator of the delay multipliers for nonlinear models.

4. *Sustained multipliers*

To properly measure the effect on the expected values of endogenous variables at time t due to unit change of an exogenous variable from time

$t-r$ to t (persisting over time), a convenient definition of sustained multipliers can be the following:

$$(4.1) \qquad \theta_{t,r} = vec\, E\left[\sum_{q=0}^{r} \frac{\partial y}{\partial x'_{t-q}} \mid x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, \alpha\right]$$

where the derivatives are computed at point $(x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1},$

$\alpha, u_t, u_{t-1}, \ldots, u_{t-r})$.

As already mentioned in section 3, all the terms of the sum in expression (4.1) are conditional on the historical values of y_{t-r-1}, so that the term related to $q=r$ is nothing but the delay-r multiplier, while the terms related to $q<r$ are generally different from the corresponding delay-q multipliers, which would be conditional on y_{t-q-1} (even if the differences are expected to be small in practical applications).

Analogously to section 3, two additional symbols need to be introduced:

$$(4.2) \qquad \bar{\theta}_{t,r} = vec\left\{\sum_{q=0}^{r}\left[\left(\frac{\partial y}{\partial x'_{t-q}}\right)_{(x_t,\, x_{t-1},\ldots,\, x_{t-r},\, y_{t-r-1},\, \alpha,\, 0,\, 0,\ldots,\, 0)}\right]\right\}$$

is the $(mn \times 1)$ column vector representing the effect on endogenous variables due to unit changes in exogenous variables, persisting over time, when the model is treated as exact (error terms are set to zero from time $t-r$);

$$(4.3) \qquad \hat{\theta}_{t,r} = vec\left\{\sum_{q=0}^{r}\left[\left(\frac{\partial y}{\partial x'_{t-q}}\right)_{(x_t,\, x_{t-1},\ldots,\, x_{t-r},\, y_{t-r-1},\, \hat{\alpha},\, 0,\, 0,\ldots,\, 0)}\right]\right\}$$

is the $(mn \times 1)$ column vector of the estimated sustained multipliers. This expression corresponds to the way in which model builders usually compute lag-r sustained multipliers at time t. Also in this case derivatives can be computed analytically, once the solution path has been obtained by means of some numerical solution method; more simply, they can be computed using finite differences.

Analogously to section 3, the following holds:

$$(4.4) \qquad plim\ \hat{\theta}_{t,r} = \bar{\theta}_{t,r}$$

and

$$(4.5) \qquad \sqrt{T}(\hat{\theta}_{t,r} - \bar{\theta}_{t,r}) =$$

$$= \sqrt{T}\, vec\left\{\sum_{q=0}^{r}\left[\left(\frac{\partial y}{\partial x'_{t-q}}\right)_{(x_t,\, x_{t-1},\ldots,\, x_{t-r},\, y_{t-r-1},\, \hat{\alpha},\, 0,\, 0,\ldots,\, 0)}\right] - \sum_{q=0}^{r}\left[\left(\frac{\partial y}{\partial x'_{t-q}}\right)_{(x_t,\, x_{t-1},\ldots,\, x_{t-r},\, y_{t-r-1},\, \alpha,\, 0,\, 0,\ldots,\, 0)}\right]\right\}$$

is asymptotically distributed as $N(0,\ G_{t,r}\Psi G'_{t,r})$, where $G_{t,r}$ is the $(mn \times s)$ matrix of the second order derivatives $(\sum_{q=0}^{r}\partial^2 y_t/\partial x_{j,t-q}\,\partial \alpha_k)$ computed at point $(x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, \alpha, 0, 0, \ldots, 0)$.

Also in this case it must be noted that not only the estimate $\hat{\theta}_{t,r}$, but also $\bar{\theta}_{t,r}$ and $\theta_{t,r}$ are random vectors, since they are all functions of the lagged endogenous variables y_{t-r-1}, so that the convergence in probability is not to a constant, but to a random variable, and the asymptotic distribution is conditional on a particular value given to the random vector y_{t-r-1}. In all the experiments the derived estimates are conditional on the historical value of the lagged endogenous variables (this is the usual way in which sustained multipliers are computed).

Computing the derivatives $(\sum_{q=0}^{r}\partial^2 y_t/\partial x_{j,t-q}\,\partial \alpha_k)$ at point $(x_t, x_{t-1}, \ldots,$ $x_{t-r}, y_{t-r-1}, \hat{\alpha}, 0, 0, \ldots, 0)$, $\hat{G}_{t,r}$, a consistent estimate of $G_{t,r}$, can be obtained; the square roots of the diagonal elements of $(\hat{G}_{t,r}\hat{\Psi}\hat{G}'_{t,r})/T$ are the estimated asymptotic standard errors of the elements of $\hat{\theta}_{t,r}$.

In nonlinear models, generally $\bar{\theta}_{t,r} \neq \theta_{t,r}$, and this inequality states the inconsistency of $\hat{\theta}_{t,r}$ as an estimate of the sustained multipliers for nonlinear models.

The next two sections discuss some algorithms to compute the asymptotic standard errors and the inconsistencies of estimated delay and sustained multipliers.

5. *An analytic simulation procedure to compute the asymptotic standard errors*

The $\hat{J}_{t,r}$ and $\hat{G}_{t,r}$ matrices can be easily computed using an analytic simulation approach. $(\partial^2 y_t/\partial x_{j,t-r}\,\partial \alpha_k)$, in fact, can be simply computed as $\Delta(\Delta\hat{y}_t/\Delta\hat{\alpha}_k)\Delta x_{j,t-r}$, using finite differences between a control solution path (dynamic simulation from time $t-r$ to time t) and solution paths obtained from disturbed values of the coefficient $\hat{\alpha}_k$ and of the exogenous variable x_j at time $t-r$ (but not disturbed in the subsequent periods). The matrix $\hat{G}_{t,r}$ can be computed in the same way, with the only difference that the shock on x_j must persist also in the subsequent periods.

In the particular case of linear models, alternative to the above method is the use of the analytical methods in Schmidt (1973), Brissimis and Gill (1978), or Gill and Brissimis (1978). However, the computer implementation of these analytical methods requires larger storage and longer computing time than the analytic simulation approach. For example, the procedure by Gill and Brissimis (1978) requires approximately 6 minutes of CPU time on an IBM/370-168 for the Klein-I model. The analytic simulation procedure described above requires, on the same computing system and for the same model, less than one second of CPU time (see Bianchi and Calzolari, 1981, for details).

Nevertheless particular care must be taken in the computation of the second order derivatives (matrices $\hat{J}_{t,r}$ and $\hat{G}_{t,r}$). They were first computed, in all experiments, from tentative values of Δx_j and $\Delta \hat{a}_k$; then these increments were gradually reduced until the computed multipliers and standard errors remained stable in the first 3-4 significant decimal digits. In most cases this experiment suggested that increments $\Delta x_j = 10^{-4} x_j$ and $\Delta \hat{a}_k = 10^{-5} \hat{a}_k$ should be adopted for most x_j and $\hat{a}_k$. Using these increments, it is also possible to evaluate the error due to cancellation of significant digits (see Stewart, 1967) in the computation of each derivative. Taking into account the accuracy to which $\hat{y}_t$ is computed and applying Stewart's formula (1967, eq. 9), the error evaluated in our experiments was never so large as to affect the first 4 significant digits.

In order to control our results, the derivatives have been computed again using central differences, which usually yield better approximations (see Stewart, 1967, pp. 75, 77), but have the disadvantage of requiring more computing time. No significant differences have been found in the first 2-3 digits of the final results.

6. *A stochastic simulation procedure to estimate the inconsistency of dynamic multipliers*

From equations (3.2), (3.3) and (3.6), it follows that the vector of inconsistencies of the estimated delay multipliers is given by

$$(6.1) \qquad i(\hat{\pi}_{t,r}) = plim\, \hat{\pi}_{t,r} - \pi_{t,r} = \bar{\pi}_{t,r} - \pi_{t,r} =$$

$$= vec\, E \left\{ \left[\frac{\partial y}{\partial x'_{t-r}} \right]_{(x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, a, 0, 0, \ldots, 0)} - \left[\frac{\partial y}{\partial x'_{t-r}} \right]_{(x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, a, u_t, u_{t-1}, \ldots, u_{t-r})} \right\}$$

where the expectation is conditional on $x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}$ and a. Approximate values for (6.1) can be obtained as sample means of replicated stochastic simulations. Starting from a consistent estimate of vector a and of the distribution parameters for structural disturbance terms (for example, their covariance matrix if the distribution is assumed multivariate normal) stochastic simulation would supply approximate values for a consistent estimate of the expectation in equation (6.1). The accuracy of the approximation, of course, increases with the number of replications and can be further enhanced by using variance reduction techniques (analogously to the case of impact multipliers in Bianchi, Calzolari, and Corsi, 1981).

Quite similar considerations hold for the sustained multipliers

$$(6.2) \qquad i(\hat{\theta}_{t,r}) = plim\, \hat{\theta}_{t,r} - \theta_{t,r} = \bar{\theta}_{t,r} - \theta_{t,r} =$$

$$= vec\, E \left\{ \sum_{q=0}^{r} \left[\left(\frac{\partial y}{\partial x'_{t-q}} \right)_{(x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, a, 0, 0, \ldots, 0)} \right] - \sum_{q=0}^{r} \left[\left(\frac{\partial y}{\partial x'_{t-q}} \right)_{(x_t, x_{t-1}, \ldots, x_{t-r}, y_{t-r-1}, a, u_t, u_{t-1}, \ldots, u_{t-r})} \right] \right\}.$$

In practice, these inconsistencies can be directly computed as sample means of replicated stochastic simulations, where the shocks to lagged exogenous variables persist over time.

The inconsistency vector has been computed by means of stochastic simulation with antithetic variates (this approach is described in detail in Calzolari, 1979, where it has been applied to compute the bias of deterministic simulation in the Klein-Goldberger model); for each exogenous variable taken into account (one or two for each model), 200 couples of replications have been performed. Inconsistencies have been computed as sample means of the 200 means of antithetic pairs. They are displayed in each table together with their experimental standard deviations (to check the degree of accuracy to which they have been computed).

7. *Some examples*

7.1. *Klein-Goldberger model*

The model on which the experiments have been performed is the revised Klein-Goldberger model described in Klein (1969). It is nonlinear and consists of 16 stochastic and 4 definitional equations, with 54 estimated coefficients; data are annual and the sample period is divided in two parts, 1929-1941 and 1947-1964.

Estimation has been performed by means of 2SLS with 4 principal components, as in Klein (1969).

Glossary

When no otherwise specified the variable is expressed in billions of 1954 dollars

Endogenous variables

Cd = Consumption of durables
Cn = Consumption of nondurables and services
R = Residential construction
H = Stock of inventories
Im = Imports
X = Gross national product
h = Index of hours worked per week, 1954 = 1
W = Wages and salaries and supplements to wages and salaries
w = Annual earnings, thousands of dollars
Pc = Corporate profits including inventory valuation adjustment
Nw = Wage and salary workers, millions
Y = Personal disposable income
p = Implicit GNP deflator, 1954 = 1
Sc = Corporate savings including inventory valuation adjustment
Π-Pc = Proprietors' income
Πr = Rental income and net interest

Exogenous variables

rd = Average discount rate at all Federal Reserve Banks, percent
T = Personal taxes plus contributions for social insurance less government and business transfer payments less interest on government debt, billions of current dollars.

Delay multipliers at 1964 of exogenous rd.

Simul. from	1964 impact	1963 delay-*1*	1962 delay-*2*	1962 sustained
Cd	-.4631e-1 (.254e-1)	-.1159 (.595e-1)	-.1714 (.880e-1)	-.3317 (.162)
Cn	-.5012e-1 (.308e-1)	-.1689 (.923e-1)	-.3264 (.178)	-.5415 (.293)
R	-.9358e-2 (.517e-2)	-.1179 (.586e-1)	-.1577 (.776e-1)	-.2844 (.138)
H	-.1233e-2 (.731e-3)	-.1025e-1 (.479e-2)	-.1800e-1 (.848e-2)	-.2935e-1 (.137e-1)
Im	-.1430e-1 (.924e-2)	-.9847e-1 (.531e-1)	-.1648 (.906e-1)	-.2784 (.151)
X	-.1038 (.600e-1)	-.8108 (.365)	-1.104 (.518)	-2.010 (.914)
h.100	-.2376e-1 (.144e-1)	-.1380 (.720e-1)	-.1126 (.722e-1)	-.2727 (.147)
W	-.5153e-1 (.300e-1)	-.4094 (.185)	-.6012 (.280)	-1.058 (.481)
w	-.5440e-3 (.382e-3)	-.4616e-2 (.255e-2)	-.1167e-1 (.637e-2)	-.1676e-1 (.900e-2)
Pc	.3671e-1 (.466e-1)	-.7675 (.395)	-.9768 (.522)	-1.668 (.904)
Nw	-.2496e-1 (.152e-1)	-.1645 (.819e-1)	-.2171 (.118)	-.4045 (.206)
Y	-.2001 (.108)	-.5226 (.246)	-.8093 (.390)	-1.523 (.697)
P	-.6519e-3 (.449e-3)	-.3989e-2 (.244e-2)	-.5605e-2 (.391e-2)	-.1032e-1 (.675e-2)
Sc	.1561e-1 (.457e-1)	-.7994 (.407)	-.9443 (.501)	-1.684 (.897)
Π	.5650e-1 (.457e-1)	-.6449 (.324)	-.8404 (.427)	-1.400 (.736)
Πr	-.1599 (.864e-1)	-.8012e-1 (.759e-1)	-.7360e-1 (.779e-1)	-.3097 (.219)

Delay multipliers at 1964 of exogenous T.

Simul. from	1964 impact	1963 delay-1	1962 delay-2	1962 sustained
Cd	-.2446 (.665e-1)	-.2545e-1 (.315e-1)	-.3983e-1 (.210e-1)	-.3018 (.775e-1)
Cn	-.2647 (.814e-1)	-.2575 (.785e-1)	-.2470 (.742e-1)	-.7509 (.232)
R	-.4942e-1 (.108e-1)	-.3124e-1 (.789e-2)	-.2445e-1 (.724e-2)	-.1024 (.228e-1)
H	-.6511e-2 (.199e-2)	-.6389e-2 (.248e-2)	-.5996e-2 (.245e-2)	-.1839e-1 (.674e-2)
Im	-.7551e-1 (.307e-1)	-.8381e-1 (.454e-1)	-.8251e-1 (.374e-1)	-.2392 (.109)
X	-.5483 (.152)	-.2636 (.145)	-.2568 (.111)	-1.039 (.391)
h.100	-.1255 (.477e-1)	.1657e-2 (.456e-1)	-.5739e-1 (.221e-1)	-.1755 (.691e-1)
W	-.2721 (.816e-1)	-.1679 (.801e-1)	-.1486 (.649e-1)	-.5720 (.214)
w	-.2873e-2 (.150e-2)	-.7651e-2 (.385e-2)	-.7953e-2 (.431e-2)	-.1814e-1 (.896e-2)
Pc	-.7240 (.205)	-.4487 (.260)	-.3846 (.143)	-1.473 (.537)
Nw	-.1318 (.489e-1)	-.1012 (.646e-1)	-.4193e-1 (.213e-1)	- 2673 (.114)
Y	-1.057 (.127)	-.2258 (.124)	-.2625 (.988e-1)	-1.505 (.327)
P	-.3443e-2 (.172e-2)	-.2870e-2 (.223e-2)	-.2706e-2 (.128e-2)	-.8955e-2 (.508e-2)
Sc	-.7446 (.222)	-.4056 (.266)	-.3079 (.102)	-1.371 (.511)
Π	-.6195 (.157)	-.3727 (.193)	-.3313 (.117)	-1.254 (.413)
Πr	.7356e-1 (.370e-1)	.4415e-1 (.462e-1)	.2460e-1 (.170e-1)	.1305 (.877e-1)

Inconsistencies of delay multipliers at 1964 of exogenous rd.

Simul. from	1964 impact	1963 delay-1	1962 delay-2	1962 sustained
Cd	.3271e-4 (.322e-5)	.5212e-4 (.159e-4)	.1991e-3 (.393e-4)	.4193e-3 (.791e-4)
Cn	.3540e-4 (.348e-5)	.9167e-4 (.191e-4)	.3081e-3 (.569e-4)	.6130e-3 (.106e-3)
R	.6608e-5 (.650e-6)	.1453e-4 (.342e-5)	.4957e-4 (.948e-5)	.1018e-3 (.183e-4)
H	.9207e-6 (.899e-7)	.2487e-5 (.504e-6)	.8413e-5 (.153e-5)	.1662e-4 (.283e-5)
Im	.6389e-5 (.712e-6)	.1307e-4 (.352e-5)	.4038e-4 (.992e-5)	.8406e-4 (.181e-4)
X	.7753e-4 (.757e-5)	.1652e-3 (.399e-4)	.5860e-3 (.110e-3)	.1192e-2 (.212e-3)
h.100	.1777e-4 (.174e-5)	.3215e-4 (.880e-5)	.1308e-3 (.237e-4)	.2610e-3 (.461e-4)
W	.3848e-4 (.376e-5)	.8796e-4 (.201e-4)	.3051e-3 (.569e-4)	.6178e-3 (.109e-3)
w	.4069e-6 (.397e-7)	.1618e-5 (.261e-6)	.4908e-5 (.896e-6)	.9321e-5 (.154e-5)
Pc	.2433e-3 (.207e-4)	.3180e-2 (.315e-3)	.6167e-2 (.721e-3)	.1012e-1 (.107e-2)
Nw	.1867e-4 (.182e-5)	.4245e-4 (.979e-5)	.1370e-3 (.269e-4)	.2866e-3 (.518e-4)
Y	.1413e-3 (.139e-4)	.2430e-3 (.697e-4)	.9047e-3 (.177e-3)	.1890e-2 (.352e-3)
P	.2293e-6 (.291e-7)	.2872e-6 (.134e-6)	.9340e-6 (.358e-6)	.2100e-5 (.627e-6)
Sc	.2804e-3 (.234e-4)	.3305e-2 (.327e-3)	.5965e-2 (.696e-3)	.1012e-1 (.106e-2)
Π	.1702e-3 (.161e-4)	.2514e-2 (.255e-3)	.4911e-2 (.592e-3)	.7976e-2 (.879e-3)
Πr	.9331e-4 (.685e-5)	-.2543e-3 (.277e-4)	-.6437e-3 (.762e-4)	-.5173e-3 (.659e-4)

Inconsistencies of delay multipliers at 1964 of exogenous T

Simul. from	1964 impact	1963 delay-1	1962 delay-2	1962 sustained
Cd	.2145e-3 (.223e-4)	.5792e-4 (.151e-4)	.1723e-3 (.233e-4)	.1212e-2 (.145e-3)
Cn	.2321e-3 (.242e-4)	.2423e-3 (.297e-4)	.3556e-3 (.412e-4)	.1998e-2 (.222e-3)
R	.4334e-4 (.451e-5)	.3209e-4 (.445e-5)	.4969e-4 (.618e-5)	.3171e-3 (.363e-4)
H	.5995e-5 (.625e-6)	.6632e-5 (.805e-6)	.9554e-5 (.112e-5)	.5340e-4 (.593e-5)
Im	.4503e-4 (.476e-5)	.4046e-4 (.590e-5)	.6711e-4 (.818e-5)	.3455e-3 (.397e-4)
X	.5049e-3 (.526e-4)	.3330e-3 (.491e-4)	.5779e-3 (.723e-4)	.3613e-2 (.415e-3)
h.100	.1157e-3 (.121e-4)	.4754e-4 (.933e-5)	.1312e-3 (.165e-4)	.7560e-3 (.877e-4)
W	.2506e-3 (.261e-4)	.1957e-3 (.268e-4)	.3091e-3 (.383e-4)	.1902e-2 (.216e-3)
w	.2649e-5 (.276e-6)	.5546e-5 (.600e-6)	.7001e-5 (.797e-6)	.3380e-4 (.362e-5)
Pc	.2313e-2 (.241e-3)	.2959e-2 (.312e-3)	.4031e-2 (.393e-3)	.1716e-1 (.170e-2)
Nw	.1215e-3 (.127e-4)	.9337e-4 (.129e-4)	.1241e-3 (.166e-4)	.8810e-3 (.102e-3)
Y	.9268e-3 (.964e-4)	.3408e-3 (.705e-4)	.8218e-3 (.107e-3)	.5572e-2 (.659e-3)
P	.1699e-5 (.187e-6)	.9159e-6 (.221e-6)	.2168e-5 (.299e-6)	.1050e-4 (.133e-5)
Sc	.2431e-2 (.251e-3)	.2829e-2 (.299e-3)	.3615e-2 (.354e-3)	.1638e-1 (.163e-2)
Π	.1945e-2 (.202e-3)	.2430e-2 (.267e-3)	.3291e-2 (.325e-3)	.1403e-1 (.142e-2)
Πr	-.3687e-3 (.375e-4)	-.4873e-3 (.546e-4)	-.5542e-3 (.546e-4)	-.2433e-2 (.237e-3)

7.2. *ISPE model of Italian economy*

The nonlinear model analyzed in this section is an annual model of the Italian economy developed by a team led by ISPE (Istituto di Studi per la Programmazione Economica).

The model, described in Sartori (1978) and Bianchi, Calzolari and Sartori (1982), consists of 19 stochastic plus 15 definitional equations; there are 75 estimated coefficients. It has been re-estimated for the period 1955-1976 using Limited information Instrumental Variables Efficient method (LIVE).

The name LIVE has been maintained since the estimation method is exactly the method by Brundy and Jorgenson (1971), but it must be recalled that the method is generally not efficient when applied to nonlinear models; the problems is discussed in Amemiya (1983) and Hatanaka (1978).

Glossary

Endogenous variables

CPNCF = Private consumption
DXML = Price deflator for exports
IFIT = Private investment
LI = Employees in industrial sector
MT = Imports of goods and services
PCL = Price deflator of private consumption
VAP = Gross output of private sector
XT = Exports of goods and services

Exogenous variables

ATI = Direct taxes rate
ERL$ = Exchange rate U.S. Dollar/Lira

Delay multipliers at 1976 of exogenous ATI.

Simul. from	1976 impact	1975 delay-1	1974 delay-2	1974 sustained
CPNCF	-.2510e+5 (.790e+4)	-.3604e+5 (.480e+4)	-.1391e+5 (.378e+4)	-.7575e+5 (.713e+4)
DXML	1.766 (.435)	.5998 (.417)	-.4673 (.397)	1.866 (.907)
IFIT	-2798. (827.)	-8794. (.227e+4)	-8089. (.238e+4)	-.1976e+5 (.458e+4)
LI	-1056. (577.)	-3185. (884.)	-2537. (988.)	-6815. (.171e+4)
MT	-.1248e+5 (.360e+4)	-.2418e+5 (.300e+4)	-.1472e+5 (.382e+4)	-.5161e+5 (.708e+4)
PCL	3.782 (.334)	.9998 (.417)	1.057 (.324)	5.731 (.834)
VAP	-.2190e+5 (.658e+4)	-.3087e+5 (.593e+4)	-8322. (.498e+4)	-.6154e+5 (.116e+5)
XT	-7523. (.275e+4)	-.1053e+5 (.420e+4)	-1828. (.315e+4)	-.1979e+5 (.849e+4)

Delay multipliers at 1976 of exogenous ERL$.

Simul. from	1976 impact	1975 delay-1	1974 delay-2	1974 sustained
CPNCF	-2.008 (.697)	-1.554 (1.21)	-.2563 (.534)	-4.156 (1.53)
DXML	.1919e-2 (.254e-3)	-.5907e-4 (.616e-4)	.1147e-3 (.777e-4)	.1951e-2 (.239e-3)
IFIT	-.4280 (.234)	.8651 (.650)	.9325e-1 (.406)	.4182 (1.11)
LI	.6233e-1 (.716e-1)	.3537 (.255)	.2619e-1 (.131)	.4050 (.398)
MT	-.1081e-1 (.694)	1.482 (1.58)	.1454 (.585)	1.356 (2.58)
PCL	.1432e-3 (.776e-4)	.1895e-4 (.650e-4)	.1459e-3 (.410e-4)	.3067e-3 (.139e-3)
VAP	.9818 (1.24)	3.331 (2.22)	-.8879 (.405)	3.109 (3.12)
XT	4.024 (1.51)	5.718 (2.33)	-.5125 (.361)	9.106 (3.51)

Inconsistencies of delay multipliers at 1976 of exogenous ATI.

Simul. from	1976 impact	1975 delay-1	1974 delay-2	1974 sustained
CPNCF	-2.210 (1.25)	-15.98 (1.59)	-21.93 (1.83)	-56.15 (5.27)
DXML	-.6135e-3 (.963e-4)	.5580e-4 (.116e-3)	.6383e-3 (.115e-3)	-.5533e-3 (.288e-3)
IFIT	-.2602e-1 (.142)	-.5412 (.418)	-3.276 (.666)	-5.048 (1.04)
LI	.1559e-1 (.670e-1)	-.4941 (.165)	-.9112 (.293)	-1.645 (.469)
MT	3.568 (.697)	2.469 (1.59)	-8.884 (1.88)	.1691 (4.28)
PCL	.3608e-3 (.678e-4)	.1245e-2 (.126e-3)	.1262e-2 (.151e-3)	.3744e-2 (.399e-3)
VAP	.2355 (1.18)	-3.680 (1.94)	-13.68 (1.98)	-25.69 (4.61)
XT	5.928 (.741)	15.33 (2.08)	2.282 (1.98)	35.07 (4.31)

Inconsistencies of delay multipliers at 1976 of exogenous ERL$.

Simul. from	1976 impact	1975 delay-1	1974 delay-2	1974 sustained
CPNCF	.5200e-2 (.605e-3)	.4984e-2 (.718e-3)	-.1976e-3 (.383e-3)	.1636e-1 (.154e-2)
DXML	-.4616e-6 (.738e-7)	-.1810e-6 (.496e-7)	-.2544e-6 (.463e-7)	-.7522e-6 (.166e-6)
IFIT	.8352e-4 (.488e-4)	-.4254e-3 (.209e-3)	-.1247e-2 (.367e-3)	-.1510e-2 (.477e-3)
LI	-.6496e-4 (.299e-4)	-.3567e-4 (.893e-4)	-.3064e-3 (.122e-3)	-.5750e-3 (.172e-3)
MT	.1197e-2 (.429e-3)	-.2494e-2 (.831e-3)	-.1065e-2 (.462e-3)	-.3584e-2 (.145e-2)
PCL	.1023e-6 (.121e-7)	.8873e-7 (.283e-7)	-.1267e-7 (.225e-7)	.1393e-6 (.639e-7)
VAP	-.2607e-3 (.463e-3)	-.2377e-2 (.122e-2)	.9291e-3 (.558e-3)	-.3716e-2 (.184e-2)
XT	-.4726e-2 (.536e-3)	-.9595e-2 (.154e-2)	.1210e-2 (.307e-3)	-.2294e-1 (.273e-2)

7.3. *Bonn model 10 of Germany (real sector)*

The sub-model, for the real economy, of the Bonn Forecasting System No. 10, used for these experiments, consists of 136 equations, 59 of which are stochastic; it includes 39 exogenous variables and 163 estimated coefficients (data are annual). For most of the equations, the estimation period is 1960-1977. For a detailed description of the model, reference should be made to Krelle (1976) and to Conrad and Kohnert (1979).

The model has been re-estimated by means of Limited information Instrumental Variables Efficient method (LIVE). The exogenous variable taken into account appears in the model only lagged.

Glossary

Endogenous variables

BP'BPGS = Bal. of paym. goods and services
C'PR = Private consumption (1970)
M'GSNO = Imports of goods and services
P'C = Price index of consumption
WR'P = Wage rate private
U'DIR = Unemployed persons
FW = Foreign workers
YDP'P = Gross domestic product private (1970)

Exogenous variable

FX'SUS = Foreign exchange rate

Delay multipliers at 1977 of exogenous FX'SUS.

Simul. from	1977 impact	1976 delay-*1*	1975 delay-*2*	1975 sustained
BP'BPGS	.0	-11.82	-3.125	-14.56
	(.0)	(1.44)	(1.51)	(1.75)
P'C	.0	.0	.6215e-3	.6215e-3
	(.0)	(.0)	(.227e-3)	(.227e-3)
WR'P	.0	.0	-.7647e-1	-.7647e-1
	(.0)	(.0)	(.355e-1)	(.355e-1)
C'PR	.0	-4.146	-2.832	-6.873
	(.0)	(.945)	(.595)	(1.05)
YDP'P	.0	-4.599	-3.148	-7.617
	(.0)	(1.28)	(1.02)	(1.51)
M'GSNO	.0	12.58	3.328	15.50
	(.0)	(1.51)	(1.61)	(1.84)
FW	.0	.0	-.3004e-1	-.3004e-1
	(.0)	(.0)	(.187e-1)	(.187e-1)
U'DIR	.0	.1114	.1039	.2115
	(.0)	(.371e-1)	(.340e-1)	(.550e-1)

Inconsistencies of delay multipliers at 1977 of exogenous FX'SUS.

Simul. from	1977 impact	1976 delay-*1*	1975 delay-*2*	1975 sustained
BP'BPGS	.0	.6781e-2	-.1304e-1	-.5363e-2
	(.0)	(.169e-2)	(.123e-2)	(.226e-2)
P'C	.0	.0	.1471e-5	.1476e-5
	(.0)	(.0)	(.214e-6)	(.215e-6)
WR'P	.0	.0	.4029e-4	.4395e-4
	(.0)	(.0)	(.257e-4)	(.259e-4)
C'PR	.0	.4410e-2	.3936e-2	.9034e-2
	(.0)	(.465e-3)	(.459e-3)	(.783e-3)
YDP'P	.0	.4325e-2	.3585e-2	.8135e-2
	(.0)	(.115e-2)	(.104e-2)	(.179e-2)
M'GSNO	.0	-.7222e-2	.1389e-1	.5712e-2
	(.0)	(.180e-2)	(.131e-2)	(.240e-2)
FW	.0	.0	.4964e-5	.5815e-5
	(.0)	(.0)	(.775e-5)	(.777e-5)
U'DIR	.0	-.3594e-4	.8145e-5	-.1951e-4
	(.0)	(.248e-4)	(.274e-4)	(.422e-4)

CARLO BIANCHI - GIORGIO CALZOLARI
PAOLO CORSI - LORENZO PANATTONI

IBM Scientific Center, Pisa

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